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A Modern Introduction to Quantum Field Theory

Michele Maggiore

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Preface

This book grew out of the notes of the course on quantum field theory that I give at the University of Geneva, for students in the fourth year.

Most courses on quantum field theory focus on teaching the student how to compute cross-sections and decay rates in particle physics. This is, and will remain, an important part of the preparation of a high-energy physicist. However, the importance and the beauty of modern quantum field theory resides also in the great power and variety of its methods and ideas. These methods are of great generality and provide a unifying language that one can apply to domains as different as particle physics, cosmology, condensed matter, statistical mechanics and critical phenomena. It is this power and generality that makes quantum field theory a fundamental tool for any theoretical physicist, independently of his/her domain of specialization, as well as, of course, for particle physics experimentalists.

In spite of the existence of many textbooks on quantum field theory, I decided to write these notes because I think that it is difficult to find a book that has a modern approach to quantum field theory, in the sense outlined above, and at the same time is written having in mind the level of fourth year students, which are being exposed for the first time to the subject.

The book is self-contained and can be covered in a two semester course, possibly skipping some of the more advanced topics. Indeed, my aim is to propose a selection of topics that can really be covered in a course, but in which the students are introduced to many modern developments of quantum field theory.

At the end of some chapters there is a Solved Problems section where some especially instructive computations are presented in great detail, in order to give a model of how one really performs non-trivial computations. More exercises, sometimes quite demanding, are provided for Chapters 1 to 8, and their solutions are discussed at the end of the book. Chapters 9, 10 and 11 are meant as a bridge toward more advanced courses at the PhD level.

A few parts which are more technical and can be skipped at a first reading are written in smaller characters.

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Notation

Our notation is the same as Peskin and Schroeder (1995). We use units $\hbar = c = 1$; their meaning and usefulness is illustrated in Section 1.2. The metric signature is

$$\eta_{\mu\nu} = (+,-,-,-)\,.$$

Indices. Greek indices take values $\mu = 0,\dots,3$, while spatial indices are denoted by Latin letters, $i,j,\dots = 1,2,3$. The totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ has $\epsilon^{0123} = +1$ (therefore $\epsilon_{0123} = -1$). Observe that, e.g. $\epsilon^{1230} = -1$ since, to recover the reference sequence 0123, the index zero has to jump three positions. Therefore $\epsilon^{\mu\nu\rho\sigma}$ is anticyclic. Repeated upper and lower Lorentz indices are summed over, e.g. $A_\mu B^\mu \equiv \sum_{\mu=0}^{3} A_\mu B^\mu$. When the equations contain only spatial indices, we will keep all indices as upper indices,[1] and we will sum over repeated upper indices; e.g. the angular momentum commutation relations are written as $[J^k, J^l] = i\epsilon^{klm} J^m$, and the totally antisymmetric tensor ϵ^{ijk} is normalized as $\epsilon^{123} = +1$. The notation $\mathbf{A}$ denotes a spatial vector whose components have upper indices, $\mathbf{A} = (A^1, A^2, A^3)$.

[1] We will never use lower spatial indices, to avoid the possible ambiguity due to the fact that in equations with only spatial indices it would be natural to use δ_{ij} to raise and lower them, while with our signature it is rather $\eta_{ij} = -\delta_{ij}$.

The partial derivative is denoted by $\partial_\mu = \partial/\partial x^\mu$ and the (flat space) d'Alambertian is $\Box = \partial_\mu\partial^\mu = \partial_0^2 - \boldsymbol{\nabla}^2$. With our choice of signature the four-momentum operator is represented on functions of the coordinates as $p^\mu = +i\partial^\mu$, so $p^0 = i\partial/\partial x^0 = i\partial/\partial t$ and $p^i = i\partial^i = -i\partial_i = -i\partial/\partial x^i$. Therefore $p^i = -i\nabla^i$ with $\nabla^i = \partial/\partial x^i = \partial_i$ or, in vector notation, $\mathbf{p} = -i\boldsymbol{\nabla}$ and $\boldsymbol{\nabla} = \partial/\partial\mathbf{x}$.

The symbol $\overleftrightarrow{\partial_\mu}$ is defined by $f \overleftrightarrow{\partial_\mu} g = f\partial_\mu g - (\partial_\mu f)g$. We also use the Feynman slash notation: for a four-vector A^μ, we define $\not{A} = A_\mu\gamma^\mu$. In particular, $\not{\partial} = \gamma^\mu\partial_\mu$.

Dirac matrices. Dirac γ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,.$$

Therefore $\gamma_0^2 = 1$ and, for each i, $(\gamma^i)^2 = -1$; γ^0 is hermitian while, for each i, γ^i is antihermitian,

$$(\gamma^0)^\dagger = \gamma^0\,, \qquad (\gamma^i)^\dagger = -\gamma^i\,,$$

or, more compactly, $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. The matrix γ^5 is defined as

$$\gamma^5 = +i\gamma^0\gamma^1\gamma^2\gamma^3\,,$$

and satisfies

$$(\gamma^5)^2 = 1\,, \qquad (\gamma^5)^\dagger = \gamma^5\,, \qquad \{\gamma^5, \gamma^\mu\} = 0\,.$$

We also define

$$\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu]\,.$$

Two particularly useful representations of the γ matrix algebra are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(here 1 denotes the 2×2 identity matrix), which is called the chiral or Weyl representation, and

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is called the ordinary, or standard, representation.

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and satisfy

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k\,.$$

We also define

$$\sigma^\mu = (1, \sigma^i)\,, \qquad \bar{\sigma}^\mu = (1, -\sigma^i)\,.$$

In the calculation of cross-sections and decay rates we often need the following traces of products of γ matrices,

$$\begin{aligned} \mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4\,\eta^{\mu\nu}\,, \\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})\,, \\ \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= -4i\epsilon^{\mu\nu\rho\sigma}\,. \end{aligned}$$

Fourier transform. The four-dimensional Fourier transform is

$$\begin{aligned} f(x) &= \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\,\tilde{f}(k)\,, \\ \tilde{f}(k) &= \int d^4x\, e^{ikx}\, f(x)\,, \end{aligned}$$

and, because of our choice of signature, the three-dimensional Fourier transform is defined as

$$\begin{aligned} f(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3}\, e^{+i\mathbf{k}\cdot\mathbf{x}}\,\tilde{f}(\mathbf{k})\,, \\ \tilde{f}(\mathbf{k}) &= \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, f(\mathbf{x})\,. \end{aligned}$$

For arbitrary n, the n-dimensional Dirac delta satisfies

$$\int d^nx\, e^{ikx} = (2\pi)^n \delta^{(n)}(k)\,.$$

Electromagnetism. The electron charge is denoted by e, and $e < 0$. As is customary in quantum field theory and particle physics, we use the Heaviside–Lorentz system of units for electromagnetism (also called *rationalized* Gaussian c.g.s. units). This means that the fine structure constant $\alpha = 1/137.035\,999\,11(46)$ is related to the electron charge by

$$\alpha = \frac{e^2}{4\pi\hbar c}\,,$$

or simply $\alpha = e^2/(4\pi)$ when we set $\hbar = c = 1$. With this definition of the unit of charge there is no factor of 4π in the Maxwell equations,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho\,, \qquad \boldsymbol{\nabla}\times\mathbf{B} - \partial_0\mathbf{E} = \mathbf{J}\,,$$

while the Coulomb potential between two static particles of charges $Q_1 = q_1 e$ and $Q_2 = q_2 e$ is

$$V(r) = \frac{Q_1 Q_2}{4\pi r} = q_1 q_2 \frac{\alpha}{r} \tag{1}$$

(where in the last equality we have used $\hbar = c = 1$), and the energy density of the electromagnetic field is

$$\varepsilon = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\,.$$

In quantum electrodynamics nowadays these conventions on the electric charge are almost universally used, but it is useful to remark that they differ from the (*unrationalized*) Gaussian units commonly used in classical electrodynamics; see, e.g. Jackson (1975) or Landau and Lifshitz, vol. II (1979), where the electron charge is rather defined so that $\alpha = e^2_{\rm unrat}/(\hbar c) \simeq 1/137$, and therefore $e_{\rm unrat} = e/\sqrt{4\pi}$. The unrationalized electric and magnetic fields, $\mathbf{E}_{\rm unrat}, \mathbf{B}_{\rm unrat}$ by definition are related to the rationalized electric and magnetic fields, $\mathbf{E}, \mathbf{B}$ by $\mathbf{E}_{\rm unrat} = \sqrt{4\pi}\,\mathbf{E}, \mathbf{B}_{\rm unrat} = \sqrt{4\pi}\,\mathbf{B}$, i.e. $A^\mu_{\rm unrat} = \sqrt{4\pi}\,A^\mu$. The form of the Lorentz force equation is therefore unchanged, since with these definitions $e\mathbf{E} = e_{\rm unrat}\mathbf{E}_{\rm unrat}$ and $e\mathbf{B} = e_{\rm unrat}\mathbf{B}_{\rm unrat}$. However, a factor 4π appears in the Maxwell equations, $\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm unrat} = 4\pi\rho_{\rm unrat}$ and $\boldsymbol{\nabla}\times\mathbf{B}_{\rm unrat} - \partial_0\mathbf{E}_{\rm unrat} = 4\pi\mathbf{J}_{\rm unrat}$; the Coulomb potential becomes $V(r) = (Q_1Q_2)_{\rm unrat}/r$, and the electromagnetic energy density becomes $\varepsilon = (\mathbf{E}^2_{\rm unrat} + \mathbf{B}^2_{\rm unrat})/(8\pi)$.

In quantum electrodynamics, since $eA^\mu = e_{\rm unrat}A^\mu_{\rm unrat}$, the interaction vertex is $-ie\gamma^\mu$ in rationalized units and $-ie_{\rm unrat}\gamma^\mu$ in unrationalized units. However, in unrationalized units the gauge field is not canonically normalized, as we see for instance from the form of the energy density. Therefore in unrationalized units the factor associated to an incoming photon in a Feynman graph becomes $\sqrt{4\pi}\epsilon^\mu$ rather than just ϵ^μ, to an outgoing photon it is $\sqrt{4\pi}\epsilon^{*\mu}$ rather than just $\epsilon^{*\mu}$, and in the photon propagator the factor $1/k^2$ becomes $4\pi/k^2$. In quantum theory it is more convenient to have a canonically normalized gauge field, which is the reason why, except in Landau and Lifshitz, vol. IV (1982), rationalized units are always used.[2]

[2]Observe that, once the result is written in terms of α, it is independent of the conventions on e, since α is always the same constant $\simeq 1/137$. For instance, the Coulomb potential between two electrons (in units $\hbar = c = 1$) is always $V(r) = \alpha/r$.

Experimental data. Unless explicitly specified otherwise, our experimental data are taken from the 2004 edition of the Review of Particle Physics of the Particle Data Group, S. Eidelman *et al.*, *Phys. Lett.* B592, 1 (2004), also available on-line at http://pdg.lbl.gov.

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Introduction

1

1.1 Overview

Quantum field theory is a synthesis of quantum mechanics and special relativity, and it is one of the great achievements of modern physics. Quantum mechanics, as formulated by Bohr, Heisenberg, Schrödinger, Pauli, Dirac, and many others, is an intrinsically non-relativistic theory. To make it consistent with special relativity, the real problem is not to find a relativistic generalization of the Schrödinger equation.[1] Wave equations, relativistic or not, cannot account for processes in which the number and the type of particles changes, as in almost all reactions of nuclear and particle physics. Even the process of an atomic transition from an excited atomic state A^* to a state A with emission of a photon, $A^* \to A + \gamma$, is in principle unaccessible to this treatment (although in this case, describing the electromagnetic field classically and the atom quantum mechanically, one can get some correct results, even if in a not very convincing manner). Furthermore, relativistic wave equations suffer from a number of pathologies, like negative-energy solutions.

A proper resolution of these difficulties implies a change of viewpoint, from wave equations, where one quantizes a single particle in an external classical potential, to quantum field theory, where one identifies the particles with the modes of a field, and quantizes the field itself. The procedure also goes under the name of second quantization.

[1] Actually, Schrödinger first found a relativistic equation, that today we call the Klein–Gordon equation. He then discarded it because it gave the wrong fine structure for the hydrogen atom, and he retained only the non-relativistic limit. See Weinberg (1995), page 4.

The methods of quantum field theory (QFT) have great generality and flexibility and are not restricted to the domain of particle physics. In a sense, field theory is a universal language, and it permeates many branches of modern research. In general, field theory is the correct language whenever we face collective phenomena, involving a large number of degrees of freedom, and this is the underlying reason for its unifying power. For example, in condensed matter the excitations in a solid are quanta of fields, and can be studied with field theoretical methods. An especially interesting example of the unifying power of QFT is given by the phenomenon of superconductivity which, expressed in the field theory language, turns out to be conceptually the same as the Higgs mechanism in particle physics. As another example we can mention that the Feynman path integral, which is a basic tool of modern quantum field theory, provides a formal analogy between field theory and statistical mechanics, which has stimulated very important exchanges between these two areas. Beside playing a crucial role for physicists, quantum field theory even plays a role in pure mathematics, and in the last 20 years the physicists' intuition stemming in particular from the path integral formulation of QFT has been at the basis of striking and unexpected advances in pure mathematics.

QFT obtains its most spectacular successes when the interaction is small and can be treated perturbatively. In quantum electrodynamics (QED) the theory can be treated order by order in the *fine structure constant* $\alpha = e^2/(4\pi\hbar c) \simeq 1/137$. Given the smallness of this parameter, a perturbative treatment is adequate in almost all situations, and the agreement between theoretical predictions and experiments can be truly spectacular. For example, the electron has a magnetic moment of modulus $g|e|\hbar/(4m_e c)$, where g is called the gyromagnetic ratio. While classical electrodynamics erroneously suggests $g = 1$, the Dirac equation gives $g = 2$, and QED predicts a small deviation from this value; the experimentally measured value is

$$\left(\frac{g-2}{2}\right)\bigg|_{\rm exp} = 0.001\ 159\ 652\ 187(4) \tag{1.1}$$

(the digit in parentheses is the experimental error on the last figure), and the theoretical prediction, computed perturbatively up to order α^4, is

$$\begin{aligned}\left(\frac{g-2}{2}\right)\bigg|_{\rm th} &= \frac{\alpha}{2\pi} - (0.328\,478\,965\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.176\,11\ldots)\left(\frac{\alpha}{\pi}\right)^3 \\ &\quad -(1.434\ldots)\left(\frac{\alpha}{\pi}\right)^4 = 0.001\ 159\ 652\ 140(5)(4)(27)\,.\end{aligned}$$

Different sources of errors on the last figures are written separately in parentheses. The theoretical error is due partly to the numerical evaluation of Feynman diagrams (there are 891 of them at order α^4!) and partly to the fact that, at this level of precision, hadronic contributions come into play. We also need to know α with sufficient accuracy; this is provided by the quantum Hall effect.

The gyromagnetic ratio has been measured very precisely also for the muon, and the accuracy of this measurement has been improved recently,[2] with the result $(g-2)/2|_{\rm exp} = 0.001\,165\,9208(6)$, and a theoretical prediction $(g-2)/2|_{\rm th} = 0.001\,165\,9181(7)$. The remaining discrepancy has aroused much interest, in the hope that it might be a signal of new physical effects, but to see whether this is actually the case requires first a better theoretical understanding of hadronic contributions, which are more difficult to compute. In any case, an agreement between theory and experiment at the level of 10 decimal figures for the electron (or eight for the muon) is spectacular, and it is among the most precise in physics.

[2] See http://www.g-2.bnl.gov/. This values updates the value reported in the 2004 edition of the Review of Particle Physics.

As we know today, QED is only a part of a larger theory. As we approach the scales of nuclear physics, i.e. length scales $r \sim 10^{-13}$ cm or energies $E \sim 200$ MeV, the existence of new interactions becomes evident: strong interactions are responsible for instance for binding together neutrons and protons into nuclei, and weak interactions are responsible for a number of decays, like the beta decay of the neutron into the proton, electron and antineutrino, $n \to pe^-\bar{\nu}_e$. A successful theory of beta decay was already proposed by Fermi in 1934. We now understand the Fermi theory as a low energy approximation to a more complete theory, that unifies the weak and electromagnetic interactions into a single conceptual framework, the electroweak theory. This theory, developed in the early 1970s, together with the fundamental theory of strong interactions, quantum chromodynamics (QCD), has such spectacular experimental successes that it now goes under the name of the Standard Model. In the last decade of the 20th century the LEP machine at CERN performed a large number of precision measurements, at the level of one part in 10^4, which are all completely reproduced by the theoretical predictions of the Standard Model. These results show that we do understand the laws of Nature down to the scale of 10^{-17} cm, i.e. four orders of magnitude below the size of a nucleus and nine orders of magnitude below the size of an atom. Part of the activity of high energy physicists nowadays is devoted to the search of physics beyond the Standard Model. The best hint for new physics presently comes from the recent experimental evidence for neutrino oscillations. These oscillations imply that neutrinos have a very small mass, whose deeper origin is suspected to be related to physics beyond the Standard Model.

The Standard Model has a beautiful theoretical structure; its discovery and development, due among others to Glashow, Weinberg, Salam and 't Hooft, requires a number of new concepts compared to QED. A detailed explanation of the Standard Model is beyond the scope of this course, but we will discuss two of its main ingredients: non-abelian gauge fields, or Yang–Mills theories, and spontaneous symmetry breaking through the Higgs mechanism.

In spite of the remarkable successes of the Standard Model, the search for the fundamental laws governing the microscopic world is still very far from being completed. In the Standard Model itself there is still a missing piece, since it predicts a particle, the Higgs boson, which plays a crucial role and which has not yet been observed. LEP, after 11 years of glorious activity, was closed in November 2000, after reaching a maximum center of mass energy of 209 GeV. The new machine, LHC, is now under construction at CERN, and together with the Tevatron collider at Fermilab aims at exploring the TeV ($= 10^3$ GeV $= 10^{12}$ eV) energy range. It is hoped that they will find the Higgs boson and that they will test theoretical ideas like supersymmetry that, if correct, are expected to give observable signals at this energy scale.

Looking much beyond the Standard Model, there is a very substantial reason for believing that we are still far from a true understanding of the fundamental laws of Nature. This is because gravity cannot be included in the conceptual schemes that we have discussed so far. General rela-